ELECTROMAGNETIC WAVE THEORY

Credits: 4

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Pre-Requisites: Vector Calculus, Engineering Physics, Applied Physics Course Objectives: This course introduces to learn the basic principles of electrostatics and Magnetostatics. It also introduces the students the fundamental theory and concepts of electromagnetic waves.

MODULE I: Electrostatics [11 Periods] Coulomb's Law, Electric Field Intensity – Fields due to Different Charge Distributions, Electric Flux Density, Gauss Law and Applications, Electric Potential, Relations Between E and V, Maxwell's Two Equations for Electrostatic Fields, Energy Density, Illustrative Problems. Convection and Conduction Currents, Dielectric Constant, Isotropic and Homogeneous Dielectrics, Continuity Equation, Relaxation Time, Poisson's and Laplace's Equations; Capacitance – Parallel Plate, Coaxial, Spherical Capacitors, Illustrative Problems.

MODULE II: Magnetostatics [9 Periods] Biot - Savart's Law, Ampere's Circuital Law and Applications, Magnetic Flux Density, Maxwell's Two Equations for Magnetostatic Fields, Magnetic Scalar and Vector Potentials, Forces due to Magnetic Fields, Ampere's Force Law, Inductances and Magnetic Energy, Illustrative Problems.

MODULE III: Time Varying Fields [10 Periods] A: Maxwell's Equations (Time Varying Fields): Faraday's Law and Transformer EMF, Inconsistency of Ampere's Law and Displacement Current Density, Maxwell's Equations in Different Final Forms and Word Statements. B: Conditions at a Boundary Surface: Dielectric-Dielectric and Dielectric-Conductor Interfaces, Illustrative Problems.

MODULE IV: EM Wave Characteristics -I [10 Periods] Wave Equations for Conducting and Perfect Dielectric Media, Uniform Plane Waves – Definition, All Relations Between E & H, Sinusoidal Variations, Wave Propagation in Lossless and Conducting Media, Conductors & Dielectrics – Characterization, Wave Propagation in Good Conductors and Good Dielectrics, Polarization, Illustrative Problems.

MODULE V: EM Wave Characteristics -II [8 Periods] Reflection and Refraction of Plane Waves – Normal and Oblique Incidences for both Perfect Conductor and Perfect Dielectrics, Brewster Angle, Critical Angle and Total Internal Reflection, Surface Impedance, Poynting Vector and Poynting Theorem – Applications, Power Loss in a Plane Conductor., Illustrative Problems.

Text Books:

1. Matthew N.O. Sadiku, "Elements of Electromagnetics", Oxford Univ. Press, 4th Edition, 2007.

2. E.C. Jordan and K.G. Balmain, "Electromagnetic Waves and Radiating Systems", PHI, 2nd Edition, 2000.

Reference Books:

1. Nathan Ida, "Engineering Electromagnetics", Springer (India) Pvt. Ltd, New Delhi, 2nd Edition, 2005.

2. William H. Hayt Jr. and John A. Buck, "Engineering Electromagnetics", TMH, 7th Edition, 2006.

E-Resources:

1. www.dannex.se/theory/1.html

2.ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-013electromagnetics-and-applications-spring-2009\

3. www.tandfonline.com/toc/uemg20/current

4. nptel.ac.in/courses/108104087

5. nptel.ac.in/courses/115101005

Course Outcomes:

At the end of the course, students will be able to:

1. Understand the principals of electrostatics using Maxwell's Equations.

2. Understand the principles of in magentostatics using Maxwell's Equation.

3. Observe the change in Maxwell's equations for time varying fields and also observe the condition at the boundary surfaces.

4. Get knowledge on propagation of EM wave in different media.

5. Get knowledge on propagation characteristics of EM wave in different media.

UNIT – I- Electrostatics

Contents

- Basics of coordinate system
- Coulomb’s Law
- Electric Field Intensity - Fields due to Different Charge Distributions
- Electric Flux Density
- Gauss Law and Applications,
- Electric Potential,
- Relations between E and V
- Maxwell’s Equations for Electrostatic Fields
- Energy Density
- Dielectric Constant
- Isotropic and Homogeneous Dielectrics,
- Continuity Equation
- Relaxation Time
- Poisson's and Laplace's Equations
- Capacitance - Parallel plate
- Problems.

INTRODUCTION

VECTOR ALGEBRA

Vector Algebra is a part of algebra that deals with the theory of vectors and vector spaces.

Most of the physical quantities are either scalar or vector quantities.

SCALAR QUANTITY:

Scalar is a number that defines magnitude. Hence a scalar quantity is defined as a quantity that has magnitude only. A scalar quantity does not point to any direction i.e. a scalar quantity has no directional component.

For example when we say, the temperature of the room is 30o C, we don‘t specify the direction.

Hence examples of scalar quantities are mass, temperature, volume, speed etc.

A scalar quantity is represented simply by a letter – A, B, T, V, S.

VECTOR QUANTITY:

A Vector has both a magnitude and a direction. Hence a vector quantity is a quantity that has both magnitude and direction.

Examples of vector quantities are force, displacement, velocity, etc.

$\vec{A}$, $\vec{V}$, $\vec{B}$, $\vec{F}$

A vector quantity is represented by a letter with an arrow over it or a bold letter.

UNIT VECTORS:

When a simple vector is divided by its own magnitude, a new vector is created known as the unit vector. A unit vector has a magnitude of one. Hence the name - unit vector.

A unit vector is always used to describe the direction of respective vector.

$$\mathbf{a}_A = \frac{\vec{A}}{|A|} \Rightarrow \vec{A} = |A|\,\mathbf{a}_A$$

Hence any vector can be written as the product of its magnitude and its unit vector. Unit Vectors along the co-ordinate directions are referred to as the base vectors. For example unit vectors along X, Y and Z directions are ax, ay and az respectively.

Position Vector / Radius Vector ($\overline{OP}$):

A Position Vector / Radius vector define the position of a point(P) in space relative to the origin(O).Hence Position vector is another way to denote a point in space.

$$\overline{OP} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$$

Displacement Vector

Displacement Vector is the displacement or the shortest distance from one point to another.

Vector Multiplication

When two vectors are multiplied the result is either a scalar or a vector depending on how they are multiplied. The two important types of vector multiplication are:

- Dot Product/Scalar Product (A.B)
- Cross product (A x B)

1. DOT PRODUCT (A. B):

Dot product of two vectors A and B is defined as:

$$\bar{A}.\bar{B} = \left|\bar{A}\right| \left|\bar{B}\right| \cos\theta_{AB}$$

Where θ_{AB} is the angle formed between A and B.
Also θ_{AB} ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$
The result of A.B is a scalar, hence dot product is also known as Scalar Product.

Properties of Dot Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then

   $\bar{A}.\bar{B}$= AxBx + AyBy + AzBz

2. $\bar{A}.\bar{B}$= |A| |B|, if $\cos\theta_{AB}=1$ which means $\theta_{AB} = 0^0$

This shows that A and B are in the same direction or we can also say that A and B are parallel to each other.

3. $\bar{A}.\bar{B}$ = - |A| |B|, if $\cos\theta_{AB}=-1$ which means $\theta_{AB} = 180^0$.

This shows that A and B are in the opposite direction or we can also say that A and B are antiparallel to each other.

4. $\bar{A}.\bar{B}$= 0, if $\cos\theta_{AB}=0$ which means $\theta_{AB} = 90^0$.

This shows that A and B are orthogonal or perpendicular to each other.

5. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x.\bar{a}_x = \bar{a}_y.\bar{a}_y = \bar{a}_z.\bar{a} = 1$$

$$\bar{a}_x.\bar{a}_y = \bar{a}_y.\bar{a}_z = \bar{a}_z.\bar{a} = 0$$

2. Cross Product (A X B):

Cross Product of two vectors A and B is given as:

$$\bar{A}X\bar{B} = |\bar{A}|\,|\bar{B}| \sin\theta_{AB}\,\bar{a}_N$$

Where θ_{AB} is the angle formed between A and B and $\bar{a}$ is a unit vector normal to both A and B. Also θ ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$

The cross product is an operation between two vectors and the output is also a vector.

Properties of Cross Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then,

$$\mathbf{A} * \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix}$$

The resultant vector is always normal to both the vectors A and B.

2. $\bar{A}X\bar{B} = 0$, if $\sin\theta_{AB} = 0$ which means $\theta_{AB} = 0^0$ or 180^0;

This shows that A and B are either parallel or antiparallel to each other.

3. $\bar{A}X\bar{B} = |\bar{A}|\,|\bar{B}|\,\bar{a}_N$, if $\sin\theta_{AB} = 0$ which means $\theta_{AB} = 90^0$.

This shows that A and B are orthogonal or perpendicular to each other.

Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x X \bar{a}_x = \bar{a}_y X \bar{a}_y = \bar{a}_z X \bar{a}_z = 0$$

$$\bar{a}_x X \bar{a}_y = \bar{a}_z\ ,\ \bar{a}_y X \bar{a}_z = \bar{a}_x\ ,\ \bar{a}_z X \bar{a}_x = \bar{a}_y$$

Coordinate transformations:

The table below gives a summary of transformations from one system to another.

Coordinate transformations

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi$ $+ \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi$ $+ \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi$ $+ A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi$ $+ A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi$ $+ A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi$ $+ A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

CO-ORDINATE SYSTEMS

Co-Ordinate system is a system of representing points in a space of given dimensions by coordinates, such as the Cartesian coordinate system or the system of celestial longitude and latitude.

In order to describe the spatial variations of the quantities, appropriate coordinate system is required. A point or vector can be represented in a curvilinear coordinate system that may be orthogonal or non-orthogonal. An orthogonal system is one in which the coordinates are mutually perpendicular to each other.

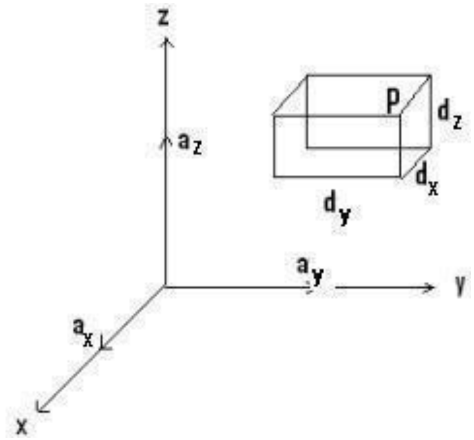
The different co-ordinate system available are:

- Cartesian or Rectangular co-ordinate system.(Example: Cube, Cuboid)
- Circular Cylindrical co-ordinate system.(Example : Cylinder)
- Spherical co-ordinate system. (Example: Sphere)

The choice depends on the geometry of the application.

A set of 3 scalar values that define position and a set of unit vectors that define direction form a co-ordinate system. The 3 scalar values used to define position are called co-ordinates. All coordinates are defined with respect to an arbitrary point called the origin.

1. Cartesian Co-ordinate System / Rectangular Co-ordinate System (x,y,z)



A Vector in Cartesian system is represented as (Ax, Ay, Az) Or

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$$

Where $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ are the unit vectors in x, y, z direction respectively.

Range of the variables:

It defines the minimum and the maximum value that x, y and z can have in Cartesian system.
-∞ ≤ x,y,z ≤ ∞

Differential Displacement / Differential Length (dl):

It is given as

$\bar{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$

Differential length for a line parallel to x, y and z axis are respectively given as:

dl = $dx\bar{a}_x$---(For a line parallel to x-axis).
dl = $dy\bar{a}_y$ ---(For a line Parallel to y-axis).
dl = $dz\bar{a}_z$ ---(For a line parallel to z-axis).

If there is a wire of length L in z-axis, then the differential length is given as dl = dz az. Similarly if the wire is in y-axis then the differential length is given as dl = dy ay.

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

$$\bar{ds} = ds\bar{a}_n$$

Where $\bar{a}_n$ is the unit vector perpendicular to the surface.

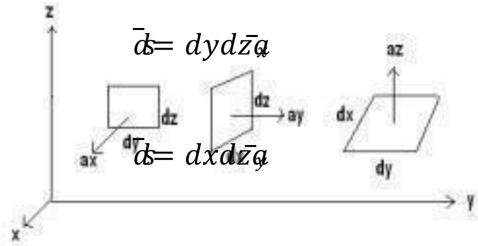
For the 1st figure,

$$\overline{ds} = dy\,dz\,\bar{a}_x$$

2nd figure,

$$\overline{ds} = dx\,dz\,\bar{a}_y$$

3rd figure,

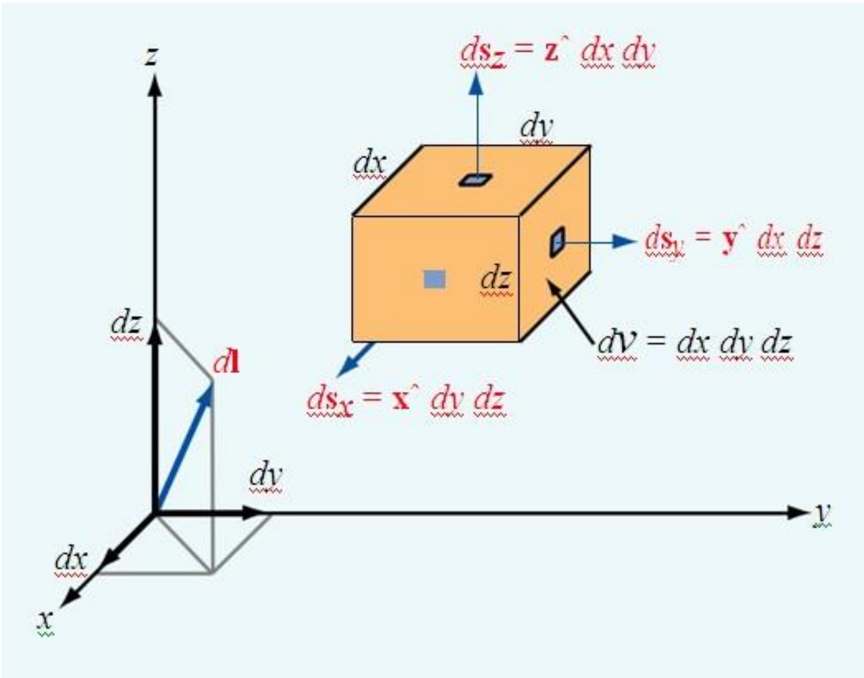


$$\overline{ds} = dx\,dy\,\bar{a}_z$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = dxdydz$$



Differential length, area, and volume in Cartesian coordinates.

2. Circular Cylindrical Co-ordinate System

A Vector in Cylindrical system is represented as (A_r, A_ϕ, A_z) or

$$\bar{A} = A_r \bar{a}_r + A_\phi \bar{a}_\phi + A_z \bar{a}_z$$

Where $\bar{a}_r$, $\bar{a}_\phi$ and $\bar{a}_z$ are the unit vectors in r, Φ and z directions respectively. The physical significance of each parameter of cylindrical coordinates:

1. The value r indicates the distance of the point from the z-axis. It is the radius of the cylinder.
2. The value Φ, also called the azimuthal angle, indicates the rotation angle around the z- axis. It is basically measured from the x axis in the x-y plane. It is measured anti clockwise.
3. The value z indicates the distance of the point from z-axis. It is the same as in the Cartesian system. In short, it is the height of the cylinder.

Range of the variables:

It defines the minimum and the maximum values of r, Φ

and z.

$0 \le r \le \infty$
$0 \le \Phi \le 2\pi$
$-\infty \le z \le \infty$

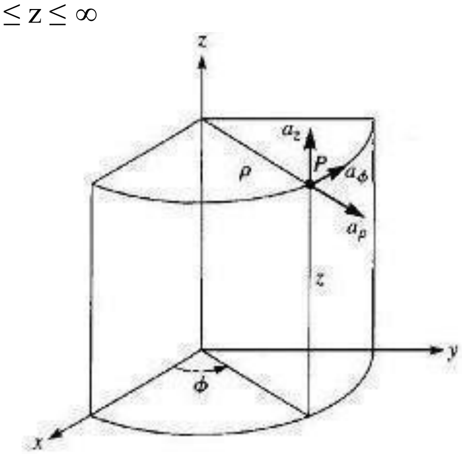


Figure shows Point P and Unit vectors in Cylindrical Co-ordinate System.

Differential Displacement / Differential Length (dl):

It is given as

$\bar{dl} = d\bar{r}a_r + rd\bar{\varphi}a_\varphi + d\bar{z}a_z$

Differential length for a line parallel to r, Φ and z axis are respectively given as: dl =

$d\bar{r}a_r$---(For a line parallel to r-direction).
dl = $rd\bar{\varphi}a_\varphi$ ---(For a line Parallel to Φ-direction). dl =
$d\bar{z}a_z$ ---(For a line parallel to z-axis).

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as

$$\bar{ds} = ds\bar{a}_n$$

Where $\bar{a}_n$ is the unit vector perpendicular to the surface.

This surface describes a circular disc. Always remember- To define a circular disk we need two parameter one distance measure and one angular measure. An angular parameter will always give a curved line or an arc.

In this case dΦ is measured in terms of change in arc.

$\bar{ds} = rdrd\bar{\varphi}a_z$

$\bar{ds} = drd\bar{z}a_\varphi$

$\bar{ds} = rdrd\bar{\varphi}a_r$

Arc is given as: Arc= radius * angle

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = rdrd\varphi dz$$

3. Spherical coordinate System:

Spherical coordinates consist of one scalar value (r), with units of distance, while the other two scalarvalues (θ, Φ) have angular units (degrees or radians).

A Vector in Spherical System is represented as (A_r ,A_Θ, A_Φ) or

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\varphi\bar{a}_\varphi$$

Where $\bar{a}_r, \bar{a}_\theta$ and $\bar{a}_\varphi$ are the unit vectors in r, θ and Φ direction respectively. The

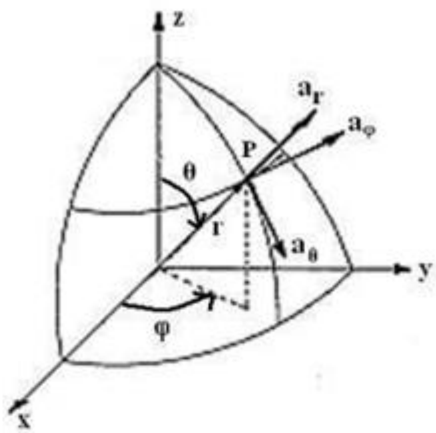
physical significance of each parameter of spherical coordinates:

1. The value r expresses the distance of the point from origin (i.e. similar to altitude). It is the radius of the sphere.
2. The angle θ is the angle formed with the z- axis (i.e. similar to latitude). It is also called the co-latitude angle. It is measured clockwise.
3. The angle Φ, also called the azimuthal angle, indicates the rotation angle around the z- axis (i.e. similar to longitude). It is basically measured from the x axis in the x-y plane. It is measured counter-clockwise.

Range of the variables:

It defines the minimum and the maximum value that r, θ and υ can have in spherical co-ordinate system.

$0 \le r \le \infty$
$0 \le \Phi \le 2\pi$
$0 \le \theta \le \pi$



Differential length:
It is given as
$d\overline{l} = dr\overline{a}_r + rd\theta\overline{a}_\theta + r \sin\theta \ d\varphi\overline{a}_\varphi$

Differential length for a line parallel to r, θ and Φ axis are respectively given as: dl = $dr\overline{a}_r$--(For a line parallel to r axis)
dl = $rd\theta\overline{a}_\theta$---(For a line parallel to θ direction)

dl = $r \sin\theta \ d\varphi\overline{a}_\varphi$ --(For a line parallel to Φ direction)

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as
$\overline{ds} = ds\overline{a}_N$
Where$\overline{a}$, is the unit vector perpendicular to the surface.
$\overline{ds} = rdrd\theta\overline{a}_\varphi$
$\overline{ds} = r^2 \sin\theta \ d\varphi d\theta\overline{a}_r$
$\overline{ds} = r \sin\theta \ drd\varphi\overline{a}_\theta$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.
$d \ = r^2 \sin\theta \ drd\varphi d\theta$

Coordinate transformations:

Coordinate transformations

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Vector relations in the three common coordinate systems.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Electrostatics:

Electrostatics is a branch of science that involves the study of various phenomena caused by electric charges that are slow-moving or even stationary. Electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics as the study of electric charges at rest.

The two important laws of electrostatics are

- Coulomb‘s Law.
- Gauss‘s Law.

Both these laws are used to find the electric field due to different charge configurations.

Coulomb‘s law is applicable in finding electric field due to any charge configurations where as Gauss‘s law is applicable only when the charge distribution is symmetrical.

Statement:

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

Mathematically,

$$F = \frac{kQ_1Q_2}{R^2} \qquad k = \frac{1}{4\pi\varepsilon_0}$$

Where k is the proportionality constant. And ε_0, is called the permittivity of free space In SI units, Q1

and Q2 are expressed in Coulombs(C) and R is in meters.
Force F is in Newton's (N)
(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore (1) $$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$$

As shown in the Figure 1 let the position vectors of the point charges Q1and Q2 are given by $\vec{r}_1$ and $\vec{r}_2$. Let $\vec{F}_{12}$ represent the force on Q1 due to charge Q2.

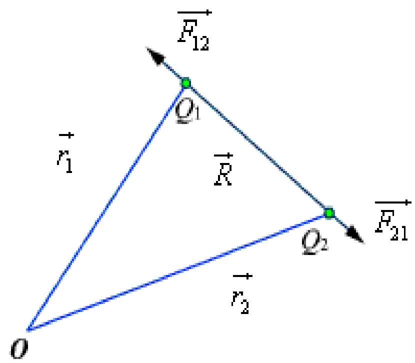


Fig 1: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r}_1 - \vec{r}_2\right| = \left|\vec{r}_2 - \vec{r}_1\right|$. We define the unit vectors as

and $$\widehat{a_{12}} = \frac{\left(\vec{r}_2 - \vec{r}_1\right)}{R} \qquad \widehat{a_{21}} = \frac{\left(\vec{r}_1 - \vec{r}_2\right)}{R}$$

$\vec{F}_{12}$ can be defined as $$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r}_2 - \vec{r}_1)}{\left|\vec{r}_2 - \vec{r}_1\right|^3}$$

Similarly the force on $Q1$ due to charge $Q2$ can be calculated and if $\vec{F}_{21}$ represents this force then we can write $\vec{F}_{21} = -\vec{F}_{12}$

Force Due to 'N' no.of point charges:

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q1$,$Q2$,. QN located respectively at the points represented by the position vectors $\vec{r}_1$, $\vec{r}_2$,. $\vec{r}_N$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i(\vec{r}-\vec{r_i})}{\left|\vec{r}-\vec{r_i}\right|^3}$$
.....

Electric Field intensity:

The electric field intensity or the electric field strength at a point is defined as the force per unit charge.

That is

or $\vec{E} = \lim_{Q\to 0}\frac{\vec{F}}{Q}$ $\quad \vec{E} = \frac{\vec{F}}{Q}$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}\,'\right|^3} \quad \text{...........(5)}$$

For a collection of *N* point charges *Q*1 ,*Q*2 ,.........*Q*N located at $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N}$, the electric field intensity at point $\vec{r}\,'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_k(\vec{r}-\vec{r_i})}{\left|\vec{r}-\vec{r_i}\right|^3} \quad (6)$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge *(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}\,'\right|^3}$$

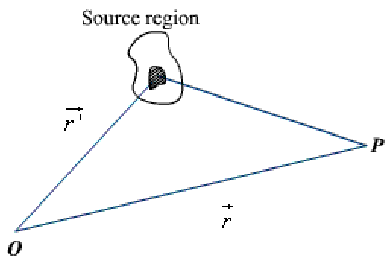


Fig 2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

...$$\overrightarrow{E(r)} = \int_V \frac{\rho(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3} dv'$$ 8)

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

...$$\overrightarrow{E(r)} = \int_L \frac{\rho_L(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3} dl'$$

....$$\overrightarrow{E(r)} = \int_S \frac{\rho_s(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3} ds'$$))

Electric flux density:

As stated earlier electric field intensity or simply ‘Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

...$\vec{D} = \varepsilon\vec{E}$................................(11)

We define the electric flux as

$$\psi = \int_S \vec{D}.d\vec{s}$$

.....................................(12)

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

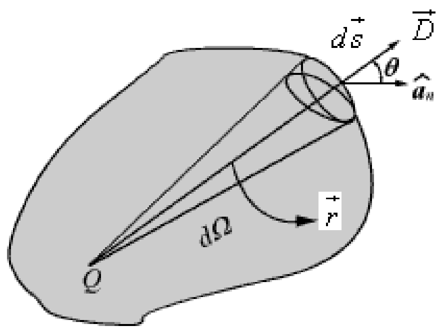


Fig 3: Gauss's Law

Let us consider a point charge *Q* located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance *r* on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \qquad \text{..............(13)}$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds\cos\theta$$

.....................................(14)

$$\frac{ds\cos\theta}{r^2} = d\Omega$$

But , is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*. Therefore we can write

For a closed surface enclosing the charge, we can write $d\psi = \frac{Q}{4\pi} d\Omega$

Which can seen to be same as what we have stated in the definition of Gauss's Law.

$$\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$$

Hence we have,

$$Q_{enc} = \oint_S \mathbf{D} \,.\, \mathbf{ds} = \int_V \rho_V \, dv$$

Applying Divergence theorem we have,

$$\oint_S \mathbf{D} \,.\, \mathbf{ds} = \int_V \nabla \cdot \mathbf{D} \, dv$$

Comparing the above two equations, we have

$$\int_V \nabla \cdot \mathbf{D} \, dv = \int_V \rho_V \, dv$$

This equation is called the 1st Maxwell's equation of electrostatics.

Application of Gauss's Law:

Gauss's law is particularly useful in computing or $\vec{E}$ $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. **An infinite line charge**

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density LC/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

....

Considering the fact that the unit normal vector to areas *S*1 and *S*3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write,

$$\rho_L l = \varepsilon_0 E.2\pi\rho l$$

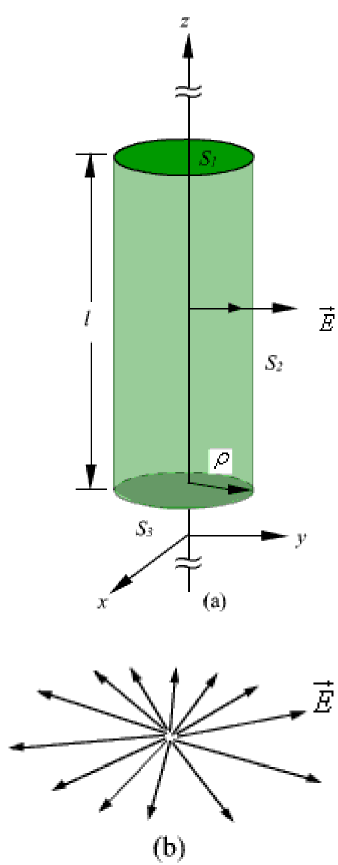


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho \qquad \text{.....................................(16)}$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering

the *x-z* plane as shown in figure 5. Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

...

$$\therefore \qquad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

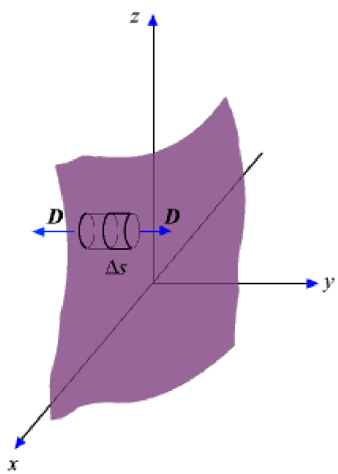


Fig 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \leq r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \quad \text{....18)}$$

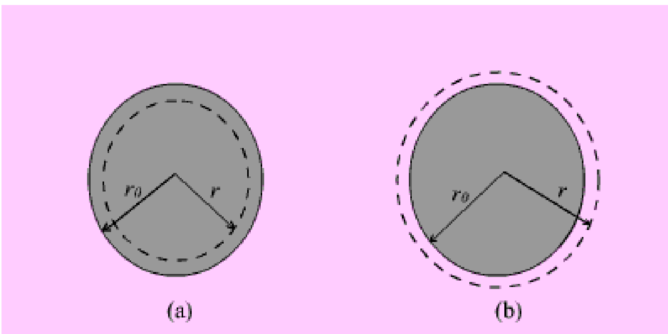


Fig 6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \leq r \leq r_0 \quad \text{.. (20)}$$

For the region $r \geq r_0$; the total enclosed charge will be

$$\text{...................................} Q_{en} = \rho_v \frac{4}{3}\pi r_0^3 \quad (21)$$

By applying Gauss's theorem,

$$\text{...}\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \geq r_0 \text{2)}$$

Electrostatic Potential:

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field. Let us suppose that we wish to move a positive test charge Δq from a point P to another point Q as shown in the Fig. 8.The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$$dW = -\Delta q \vec{E}.d\vec{l}$$ .... (23)

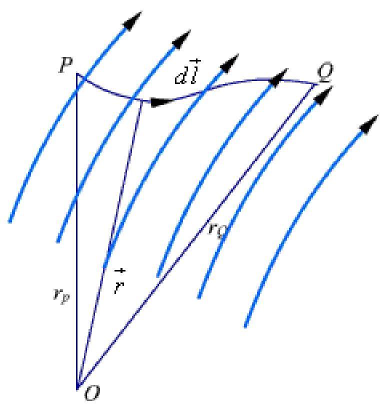


Fig 8: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l}$$ .... (24)

The potential difference between two points P and Q , VPQ, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l}$$ .... (25)

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.
We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as Volts.
Let us consider a point charge Q as shown in the Fig. 9.

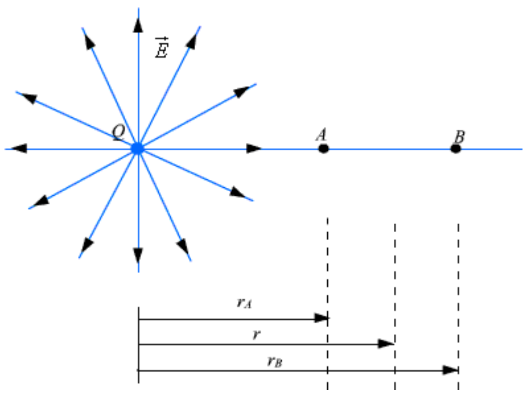


Fig 9: Electrostatic Potential calculation for a point charge

Further consider the two points A and B as shown in the Fig. 9. Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

..................($V_{BA} = -\int_B^A \vec{E}\cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$

It is customary to choose the potential to be zero at infinity. Thus potential at any point (rA = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. rB = 0).

.... $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

Or, in other words,

.................................$V = -\int_{\infty}^{r} E.dl$

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 10.

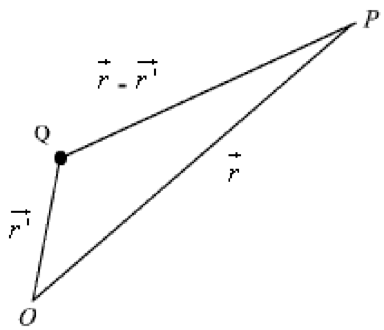


Fig 10: Electrostatic Potential due a Displaced Charge The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\left|\vec{r} - \vec{r}'\right|}$$ ....(29)

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also. Let us first consider N point charges Q1, Q2 ,. QN located at points with position vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_N}$ The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r} - \vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r} - \vec{r_2}\right|} + \ldots\ldots \frac{Q_N}{\left|\vec{r} - \vec{r_N}\right|}\right)$$ ....

OR

.... $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r} - \vec{r_n}\right|}$$ b)

For continuous charge distribution, we replace point charges Qn by corresponding charge elements

or $\rho_L dl$ or $\rho_S ds$, $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge, (31) $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{\left|\vec{r}-\vec{r}_n\right|}$$

For surface charge, (32) $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{\left|\vec{r}-\vec{r}_n\right|}$$

For volume charge, (33) $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{\left|\vec{r}-\vec{r}_n\right|}$$

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$
..................................(34)

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$V = -\int \vec{E}\cdot d\vec{l} + C$ ……………….. (35)

The potential difference is however independent of the choice of reference.

.......................(36) $$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W}{Q}$$

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P1 to P2 in one path and then from point P2 back to P1 over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable; the work must have to be independent of path and depends on the initial and final positions.

Second Maxwell's Equation of Electrostatics:

Since the potential difference is independent of the paths taken, VAB = - VBA , and over a closed path,

$V_{BA} + V_{AB} = \oint \vec{E}\cdot d\vec{l} = 0$ ....(37)

Applying Stokes's theorem, we can write:

...$\oint \vec{E}\cdot d\vec{l} = \int_S \left(\nabla\times\vec{E}\right)\cdot d\vec{s} = 0$

fr... electrostatic field,

......................(39) $$\nabla\times\vec{E} = 0$$

Any vector field that satisfies is called an irrotational field. From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial x}dz = -\vec{E}\cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)\cdot\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E}\cdot d\vec{l}$$

.................................(40)

$$\nabla V\cdot d\vec{l} = -\vec{E}\cdot d\vec{l}$$

from which we obtain,

....$\vec{E} = -\nabla V$....................... (41)

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of V. This gives us another method of computing the electric field , i.e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function V contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla\times\vec{E}=0$.

Work done in moving a point charge in an electrostatic field:

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q1, Q2,. , QN are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q1 is zero. Q2 is brought in the presence of the field of Q1, the work done W1= Q2V21 where V21 is the potential at the location of Q2 due to Q1. Proceeding in this manner, we can write, the total work done (45)

Had the charges been brought in the reverse order,

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots\ldots\ldots + (V_{N1}Q_N + \ldots\ldots + V_{N(N-1)}Q_N)$$

$$W = (V_{1N}Q_1 + \ldots\ldots + V_{12}Q_1) + \ldots\ldots\ldots\ldots\ldots + (V_{(N-2)(N-1)}Q_{N-2} + V_{(N-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$$

................(46)

Therefore,

$$2W = (V_{1N} + V_{1(N-1)} + \ldots\ldots + V_{12})Q_1 + (V_{2N} + V_{2(N-1)} + \ldots\ldots + V_{23} + V_{21})Q_1\ldots\ldots$$

.... $\ldots\ldots\ldots\ldots\ldots + (V_{N1} + \ldots\ldots V_{N2} + V_{N(N-1)})Q_N$

............(47)

Here VIJ represent voltage at the Ith charge location due to Jth charge. Therefore,

Or, $2W = V_1Q_1 + \ldots\ldots\ldots\ldots + V_NQ_N = \sum_{I=1}^{N}V_IQ_I \qquad W = \frac{1}{2}\sum_{I=1}^{N}V_IQ_I$

If instead of discrete charges, we now have a distribution of charges over a volume v then we can write,

.... $W = \frac{1}{2}\int_v V\rho_v dv$

where ρ is the volume charge density and V represents the potential function.

$$\rho_v = \nabla \cdot \vec{D}$$

Since, , we can write

$$W = \frac{1}{2}\int_v \left(\nabla \cdot \vec{D}\right) V dv$$

....(50)

$$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$$

Using the vector identity,

, we can write

$$W = \frac{1}{2}\int_v \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right) dv$$

$$= \frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v \left(\vec{D}\cdot\nabla V\right) dv \qquad \text{................(51)}$$

$$\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$$

In the expression $\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, the term V $\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r^2. Hence the integral term varies at least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \to \infty$) the integral term tends to zero.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_v \left(\vec{D}.\nabla V\right) dv = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right) dv = \frac{1}{2}\int_v \left(\varepsilon E^2\right) dv = \int_v w_e dv \qquad \text{................(52)}$$

$w_e = \frac{1}{2}\varepsilon E^2$ is called the energy density in the electrostatic field.

Maxwell's first law:

Statement: The following Electrostatic Field equations will be developed in this section:

Integral form	Differential forms
$\oint_{\text{Surface}} \boldsymbol{D} \bullet d\boldsymbol{a} = \int_{\text{Volume}} \rho\, dv.$	$\nabla \bullet \boldsymbol{D} = \rho.$ $\text{div } \boldsymbol{D} = \rho$

Maxwell's first equation is based on Gauss' law of electrostatics published in 1832, wherein Gauss established the relationship between static electric charges and their accompanying static fields.

The above integral equation states that the electric flux through a closed surface area is equal to the total charge enclosed.

The differential form of the equation states that the divergence or outward flow of electric flux from a point is equal to the volume charge density at that point.

Using vector identity we can write, (55) $\varepsilon\nabla\cdot\nabla V + \nabla V\cdot\nabla\varepsilon = -\rho_v$

For a simple homogeneous medium, is constant ε and . Therefore, $\nabla\varepsilon = 0$

................$\nabla\cdot\nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon}$

This equation is known as Poisson's equation. Here we have introduced a new operator ∇^2, (del square), called the Laplacian operator. In Cartesian coordinates,

...............($\nabla^2 V = \nabla\cdot\nabla V = (\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z)\cdot(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z)$

Therefore,

...............($\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon}$

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$ ....(59)

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$ ....(60)

At points in simple media, where no free charge is present, Poisson's equation reduces to

...................................(61) $\nabla^2 V = 0$

Which is known as Laplace's equation.

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found hroughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

Properties of Materials and Steady Electric Current:

Electric field can not only exist in free space and vacuum but also in any material medium. When an electric field is applied to the material, the material will modify the electric field either by strengthening it or weakening it, depending on what kind of material it is.

Materials are classified into 3 groups based on conductivity / electrical property:

- Conductors (Metals like Copper, Aluminum, etc.) have high conductivity ($\sigma >> 1$).
- Insulators / Dielectric (Vacuum, Glass, Rubber, etc.) have low conductivity ($\sigma << 1$).
- Semiconductors (Silicon, Germanium, etc.) have intermediate conductivity.

Conductivity (σ) is a measure of the ability of the material to conduct electricity. It is the reciprocal of resistivity (ρ). Units of conductivity are Siemens/meter and mho.

The basic difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current. Conductors have a large amount of free electrons where as insulators have only a few number ofelectrons for conduction of current. Most of the conductors obey ohm's law. Such conductors are also called ohmic conductors. Due to the movement of free charges, several types of electric current can be caused.

The different types of electric current are:

- Conduction Current.

- Convection Current.
- Displacement Current.

Electric current:

Electric current (I) defines the rate at which the net charge passes through a wire of cross sectional surface area S. Mathematically,

If a net charge ΔQ moves across surface S in some small amount of time Δt, electric current(I) is defined as:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

How fast or how speed the charges will move depends on the nature of the material medium.

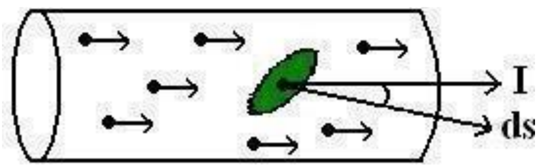
Current density:

Current density (J) is defined as current ΔI flowing through surface ΔS.

Imagine surface area ΔS inside a conductor at right angles to the flow of current. As the area approaches zero, the current density at a point is defined as:

$$J = \lim_{\Delta s \to 0} \frac{\Delta I}{\Delta S}$$

The above equation is applicable only when current density (J) is normal to the surface.
In case if current density(J) is not perpendicular to the surface, consider a small area ds of the conductor at an angle θ to the flow of current as shown:



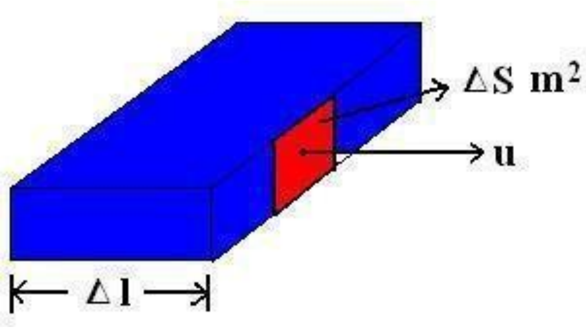
In this case current flowing through the area is given as:

$dI = J\ dS \cos\theta = J\ .\ dS$ and $I = \int_s \overline{J.\ ds}$

Where angle θ is the angle between the normal to the area and direction of the current. From the above equation it's clear that electric current is a scalar quantity.

CONVECTION CURRENT DENSITY:

Convection current occurs in insulators or dielectrics such as liquid, vacuum and rarified gas. Convection current results from motion of electrons or ions in an insulating medium. Since convection current doesn't involve conductors, hence it does not satisfy ohm's law. Consider a filament where there is a flow of charge ρv at a velocity u = uy ay.



$$\Delta I = \frac{\Delta Q}{\Delta t}$$

But we know $\boxed{\Delta Q = \rho_V \Delta V}$

Hence

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_V \Delta V}{\Delta t} = \rho_V \Delta S \frac{\Delta l}{\Delta t}$$

$$= \rho_V \Delta S\, u_y$$

Again, we also know that $\boxed{J_y = \frac{\Delta I}{\Delta S}}$

Hence $$J_y = \frac{\Delta I}{\Delta S} = \rho_V u_y$$

- Hence the current is given as:

Where uy is the velocity of the moving electron or ion and ρ_v is the free volume charge density.

- Hence the convection current density in general is given as:

$J = \rho_v u$

Conduction Current Density:

Conduction current occurs in conductors where there are a large number of free electrons. Conduction current occurs due to the drift motion of electrons (charge carriers). Conduction current obeys ohm's law.

When an external electric field is applied to a metallic conductor, conduction current occurs due to the drift of electrons.

The charge inside the conductor experiences a force due to the electric field and hence should accelerate but due to continuous collision with atomic lattice, their velocity is reduced. The net effect is that the electrons moves or drifts with an average velocity called the drift
velocity (υd) which is proportional to the applied electric field (E).

Hence according to Newton's law, if an electron with a mass m is moving in an electric field E with anaverage drift velocity υd, the the average change in momentum of the free electron must be equal to the applied force (F = - e E).

$$\frac{m\upsilon_d}{\tau} = -eE$$

where τ is the average time interval between collision.

$$\upsilon_d = \left(-\frac{e\tau}{m}\right)E$$

The drift velocity per unit applied electric field is called the mobility of electrons (μe). υd = - μe E
where μe is defined as:

$$\mu_e = \left(-\frac{e\tau}{m}\right)$$

Consider a conducting wire in which charges subjected to an electric field are moving with drift velocity υd.
Say there are Ne free electrons per cubic meter of conductor, then the free volume charge density(ρv)within the wire is
ρv= - e Ne
The charge ΔQ is given as:
ΔQ = ρv ΔV = - e Ne ΔS Δl = - e Ne ΔS υd Δt

- The incremental current is thus given as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = -N_e e \Delta S \upsilon_d$$

Now since $\boxed{\upsilon_d = -\mu_e E}$

Therefore

$$\Delta I = N_e e \Delta S \mu_e E$$

The conduction current density is thus defined as:

$$J_c = \frac{\Delta I}{\Delta S} = N_e e \mu_e E = \sigma E$$

where σ is the conductivity of the material.

The above equation is known as the Ohm‘s law in point form and is valid at every point in space.
In a semiconductor, current flow is due to the movement of both electrons and holes, hence conductivity is given as:
σ = (Ne μe + Nh μh)e
DIELECTRC CONSTANT:
It is also known as Relative permittivity.

If two charges q 1 and q 2 are separated from each other by a small distance r. Then by using the coulombs law of forces the equation formed will be

$$\mathbf{F}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

In the above equation ε_0 is the electrical permittivity or you can say it, Dielectric constant.
If we repeat the above case with only one change i.e. only change in the separation medium between the charges. Here some material medium must be used. Then the equation formed will be.

$$\mathbf{F}_m = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

Now after division of above two equations

$$\frac{\mathbf{F}_o}{\mathbf{F}_m} = \frac{\varepsilon}{\varepsilon_o} = \varepsilon_r \text{ Or } k$$

In the above figure
$\varepsilon_\mathbf{r}$ is the Relative Permittivity. Again one thing to notice is that the dielectric constant is represented by the symbol (K) but permittivity by the symbol $\varepsilon_\mathbf{r}$

CONTINUITY EQUATION:

The continuity equation is derived from two of Maxwell's equations. It states that the divergence of the current density is equal to the negative rate of change of the charge density,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Derivation

One of Maxwell's equations, Ampère's law, states that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking the divergence of both sides results in

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t},$$

but the divergence of a curl is zero, so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = 0. \qquad (1)$$

Another one of Maxwell's equations, Gauss's law, states that

$$\nabla \cdot \mathbf{D} = \rho.$$

Substitute this into equation (1) to obtain

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation.

1.13 RELAXATION TIME:

- Let us consider that a charge is introduced at some interior point of a given material (conductor or dielectric).
- From, continuity of current equation, we have

  $$\overline{J} = \frac{-rf_v}{rt} ----(1)$$

- We have, the point form of Ohm's law as,

  $$\overline{J} = 6\overline{E} ---(2)$$

- From Gauss's law, we have,

  $$\nabla \overline{D} = f_v \Rightarrow \in \nabla.\overline{E} = f_v \left[\because \overline{D} = \in.\overline{E}\right]$$

  $$\therefore \nabla.\overline{E} = \frac{f_v}{\in} -----(1)$$

- Substitute equations (2) and (3) in equation (1), we get

  $$\nabla.6\overline{E}f = 6.\nabla.\overline{E} = 6.\frac{f_v}{\in} = \frac{-\partial f_v}{\partial t}$$

  $$\Rightarrow \frac{\partial f_v}{\partial t} + \frac{6}{\in}.f_v = 0 -----(4)$$

- The above equation is a homogeneous linear ordinary differential equation. By separating variable in eq (4), we get,

  $$\frac{\partial f_v}{\partial t} = \frac{-6}{\in}.f_v$$

  $$\Rightarrow \frac{\partial f_v}{\partial t} = \frac{-6}{\in}.\partial t$$

- Now integrate on both sides of above equation

  $$\int \frac{\partial f_v}{\partial t} = -\frac{6}{\in}.\int \partial t$$

  $$\Rightarrow \ln f_v = -\frac{6}{\in}t + \ln f_{v0}$$

  Where $\ln p_{v0}$ is a constant of integration.

  Thus,

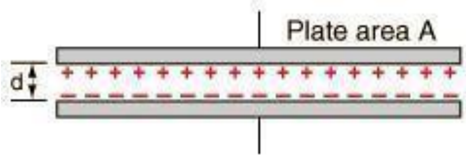
  $$\boxed{f_v = f_{v0}e^{-t/Tr}} -----(5)$$

  $$\boxed{T_r = \frac{\in}{6}}$$

- In eq (5), f_{v0} is the initial charge density (i.e fv at t=0).
- We can see from the equation that as a result of introducing charge at some interior point of the material there is a decay of volume charge density f_v.
- The time constant "T_r" is known as the relaxation time or rearrangement time.
- Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} = 36.8 percent f its initial value.
- The relation time is short for good conductors and long for good dielectrics.

Capacitance:

The capacitance of a set of charged parallel plates is increased by the insertion of adielectric material. The capacitance is inversely proportional to the electric field between the plates, and the presence of the dielectric reduces the effective electric field. The dielectric is characterized by a dielectric constant k, and the capacitance is multiplied by that factor.
Parallel Plate Capacitor



$$C = \frac{\varepsilon A}{d} = \frac{k\varepsilon_0 A}{d}$$

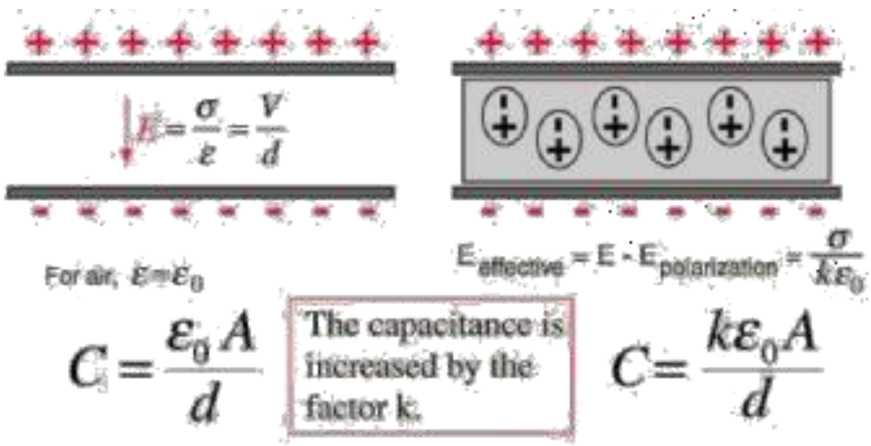
Show

The capacitance of flat, parallel metallic plates of area A and separation d is given by the expression above where:

$$\varepsilon_0 = 8.854 \ x \ 10^{-12} \ \ F/m$$

= permittivity of space and

k = relative permittivity of the dielectric material between the plates. k=1 for free space, k>1 for all media, approximately =1 for air.
The Farad, F, is the SI unit for capacitance and from the definition of capacitance is seen to be equal to a Coulomb/Volt.



Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 1. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

.... $\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2}$

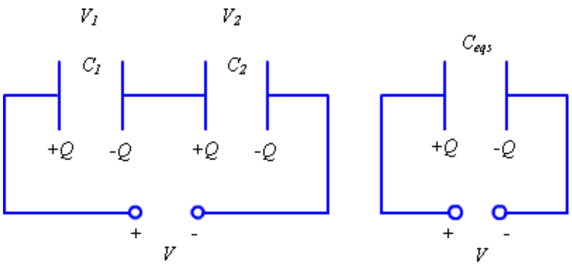


Fig 1.: Series Connection of Capacitors

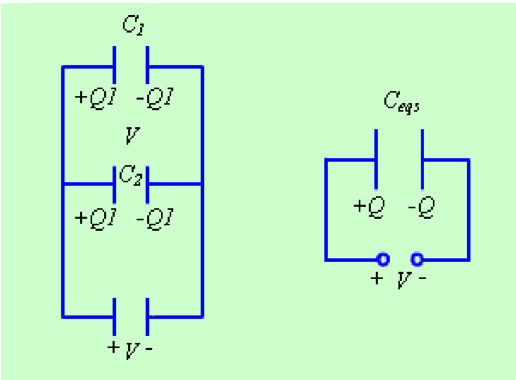


Fig 2: Parallel Connection of Capacitors

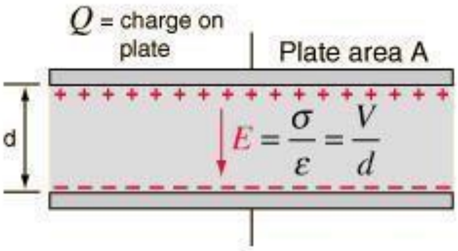
The same approach may be extended to more than two capacitors connected in series. Parallel Case: For the parallel case, the voltages across the capacitors are the same.

The total charge $Q = Q_1 + Q_2 = C_1 V + C_2 V$

Therefore, $C_{eqp} = \frac{Q}{V} = C_1 + C_2$

Capacitance of Parallel Plates:

The … el plates is given by



$$E = \frac{\sigma}{\varepsilon} \quad \text{where} \quad \begin{matrix} \sigma = \text{charge density} \\ \varepsilon = \text{permittivity} \end{matrix}$$

$$\text{and } \sigma = \frac{Q}{A}$$

The voltage difference between the two plates can be expressed in terms of the wor4k3done on a positive test charge q when it moves from the positive to the negative plate.

$$V = \frac{work\ \ done}{charge} = \frac{Fd}{q} = Ed$$

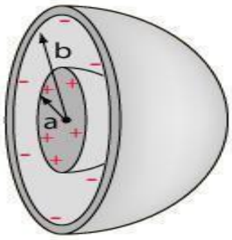
It then follows from the definition of capacitance that

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q\varepsilon}{\sigma d} = \frac{QA\varepsilon}{Qd} = \frac{A\varepsilon}{d}$$

Spherical Capacitor:

The capacitance for spherical or cylindrical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each.
By applying Gauss' law to an charged conducting sphere, the electric field outside it is **found to be**

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



The voltage between the spheres can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

From the definition of capacitance, the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Isolated Sphere Capacitor:

An isolated charged conducting sphere has capacitance. Applications for such a capacitor may not be immediately evident, but it does illustrate that a charged sphere has stored some energy as a result of being charged. Taking the concentric sphere capacitance expression:

$$C = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

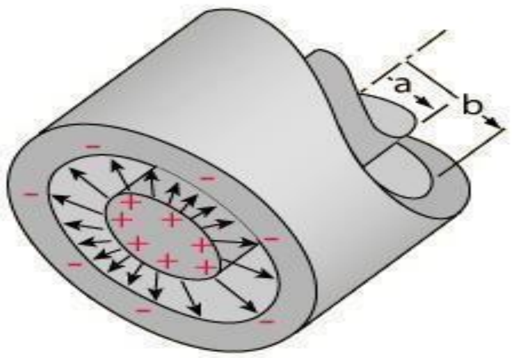
$$C = 4\pi\varepsilon_0 R$$

and taking the limits gives $a \to R$ and $b \to \infty$

Further confirmation of this comes from examining the potential of a charged conducting sphere:

$$V = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{so at the surface} \quad C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

Cylindrical Capacitor:
For a cylindrical geometry like a coaxial cable, the capacitance is usually stated as a capacitance per unit length. The charge resides on the outer surface of the inner conductor and the inner wall of the outer conductor. The capacitance expression is



$$\frac{C}{L} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

The capacitance for cylindrical orspherical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each. By applying Gauss' law to an infinite cylinder in a vacuum, the electric field outside a charged cylinder is found to be

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

The voltage between the cylinders can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{b}{a}\right] \qquad \frac{C}{L} = \frac{\lambda}{\Delta V} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

From the definition of capacitance and including the case where the volume is filled by a dielectric of dielectric constant k, the capacitance per unit length is defined above.

UNIT-II

MAGNETOSTATICS

- Biot - Savart's Law
- Ampere's Circuital Law and Applications
- Magnetic Flux Density
- Maxwell’s Equations for Magnetostatic Fields
- Magnetic Scalar and Vector Potentials
- Forces due to Magnetic Fields
- Ampere's Force Law
- Inductance and Magnetic Energy
- Illustrative Problems

Introduction:

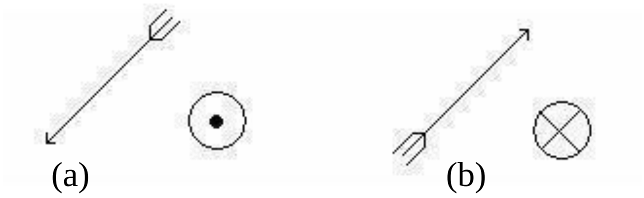
In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated.

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later.

There are two major laws governing the magneto static fields are:

- Biot-Savart Law
- Ampere's Law

Usually, the magnetic field intensity is represented by the vector $\vec{H}$. It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 2.1.



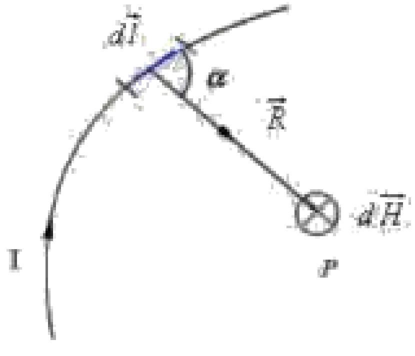
$\vec{H}$

H(or I) out of the page **H**(or I) into the page

Fig. Representation of magnetic field (or current)

Biot- Savart's Law:

This law relates the magnetic field intensity dH produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig.



The magnetic field intensity at P $d\vec{H}$ can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$

$$dH = \frac{IdlSin\alpha}{4\pi R^2}$$

where $R = |\vec{R}|$ is the distance of the current element from the point P.

The value of the constant of proportionality 'K' depends upon a property called permeability of the medium around the conductor. Permeability is represented by symbol 'm' and the constant 'K' is expressed in terms of 'm' as

Thus

$$dB = \frac{\mu}{4\pi}\frac{I\,dl\ \sin\theta}{r^2}$$

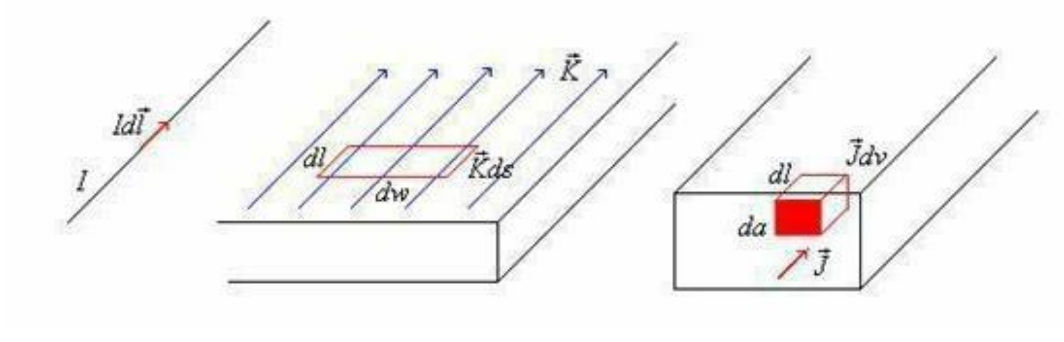
Magnetic field 'B' is a vector and unless we give the direction of 'dB', its description is not complete. Its direction is found to be perpendicular to the plane of 'r' and 'dl'.

If we assign the direction of the current 'I' to the length element 'dl', the vector product dl x r has magnitude r dl sinq and direction perpendicular to 'r' and 'dl'.

Hence, Biot–Savart law can be stated in vector form to give both the magnitude as well as direction of magnetic field due to a current element as

$$\overrightarrow{dB} = \frac{\mu}{4\pi}\frac{I\,(\overrightarrow{dl} \times \vec{r})}{r^3}$$

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 2.3.



Line Current Surface Current Volume Current

Fig. 2.3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m2) we can write:

$$I d\vec{l} = \vec{K} ds = \vec{J} dv$$

(It may be noted that $I = K dw = J da$)

Employing Biot -Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions as

$$\vec{H} = \int_l \frac{I d\vec{l} \times \vec{R}}{4\pi R^3}$$ .... for line current...........................

$$\vec{H} = \int_s \frac{K d\vec{s} \times \vec{R}}{4\pi R^3}$$ .... for surface current ....

$$\vec{H} = \int_v \frac{\vec{J} dv \times \vec{R}}{4\pi R^3}$$ .... for volume current.....................

***H* Due to infinitely long straight conductor:**

We consider a finite length of a conductor carrying a current $\vec{I}$ placed along z-axis as shown in the Fig 2.4. We determine the magnetic field at point P due to this current carrying conductor.

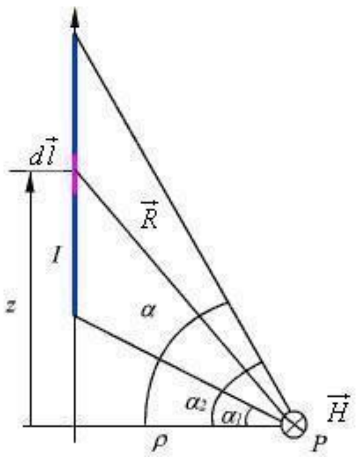


Fig. 2.4: Field at a point P due to a finite length current carrying conductor With reference to Fig. 2.4, we find that

$$d\vec{l} = dz\,\hat{a}_z \quad and \quad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

Applying Biot - Savart's law for the current element $\vec{I}\, d\vec{l}$ We can write,

$$\overrightarrow{dH} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\rho dz \hat{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}}$$

Substituting $\frac{z}{\rho} = \tan\alpha$ we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi} \frac{\rho^2 \sec^2 \alpha d\alpha}{\rho^3 \sec^3 \alpha} \hat{a}_\phi = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\phi$$

We find that, for an infinitely long conductor carrying a current I , and $\alpha_1 = -90^0$ $\alpha_2 = 90^0$ Therefore

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

Ampere's Circuital Law:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla\times\vec{H}.d\vec{s}$$
$$\therefore \int_S \nabla\times\vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s}$$
$$\therefore \nabla\times\vec{H} = \vec{J}$$

Which is the Ampere's circuital law in the point form and Maxwell's equation for magneto static fields.

Applications of Ampere's circuital law:

1. It is used to find $\vec{H}$ and $\vec{B}$ due to any type of current distribution.
2. If $\vec{H}$ or $\vec{B}$ is known then it is also used to find current enclosed by any closed path.

We illustrate the application of Ampere's Law with some examples.

$\vec{H}$ Due to infinitely long straight conductor :(using Ampere's circuital law)

We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 2.5. Using Ampere's Law, we consider the close path to be a circle of radius ρ as shown in the Fig. 4.5.

If we consider a small current element $Id\vec{l}(= Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(= \rho\hat{a}_\rho)$. Therefore only component of **H** that will be present is, $\vec{H}$ i.e., $\vec{H} = H_\phi\hat{a}_\phi$

By applying Ampere's law we can write,

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \int_0^{2\pi} H_\phi\rho d\phi = H_\phi\rho 2\pi = I$$

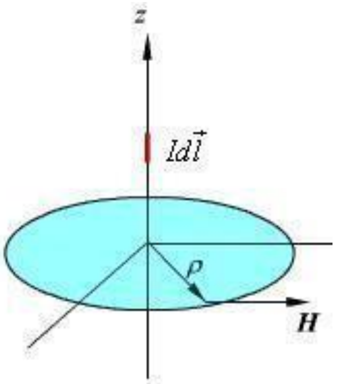


Fig. Magnetic field due to an infinite thin current carrying conductor

***H* Due to infinitely long coaxial conductor :(using Ampere's circuital law)**

We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 2.6. We compute the magnetic field as a function of ρ as follows:

In the region $0 \le \rho \le R_1$

$$I_{enc} = I\frac{\rho^2}{R_1^2}$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2}$$

In the region $R_1 \le \rho \le R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

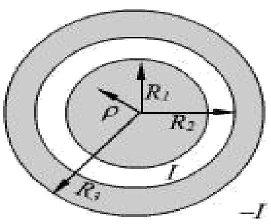


Fig. 2.6: Coaxial conductor carrying equal and opposite currents in the region

$R_2 \le \rho \le R_3$

$$H_\phi = \frac{I}{2\pi\rho}\frac{R_3^2-\rho^2}{R_3^2-R_2^2}$$

In the region $\rho > R_3$

$$I_{enc} = 0 \qquad H_\phi = 0$$

Magnetic Flux Density:

In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where μ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 = 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m 2 .

The magnetic flux density through a surface is given by:

$$\psi = \int_S \vec{B}.d\vec{s} \quad \text{Wb}$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 6 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

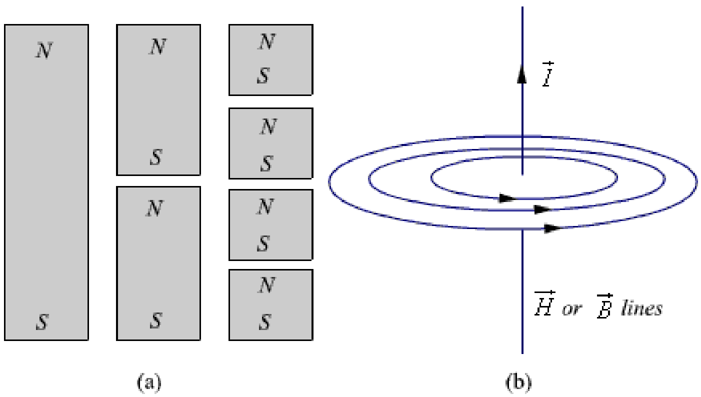


Fig. 6: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

Maxwell's 2nd equation for static magnetic fields:

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 6 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

..................$\oint_S \vec{B}.d\vec{s} = 0$...in integral form

which is the Gauss's law for the magnetic field. By applying divergence theorem, we can write:

$$\oint_S \vec{B}.d\vec{s} = \int_V \nabla.\vec{B}dv = 0$$

Hence, $\nabla.\vec{B} = 0$ in point/differential form which is the Gauss's law for the magnetic field in point form.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$$\vec{H} = -\nabla V_m$$

From Ampere's law , we know that $\nabla \times \vec{H} = \vec{J}$

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$

But using vector identity, $\nabla \times (\nabla V) = 0$, we find that $\vec{H} = -\nabla V_m$ is valid only where. $\vec{J} = 0$

Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, Vm in general is not a single valued function of position. This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 7.

In the regi, $a < \rho < b$ $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

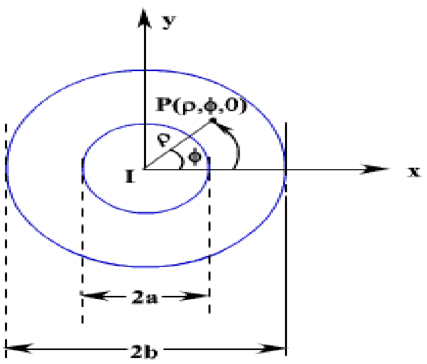


Fig. 7: Cross Section of a Coaxial Line

If Vm is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

If we set Vm = 0 at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but Vm this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of Vm keeps changing as we complete additional laps to pass through the same point. We introduced Vm analogous to electostatic potential V.

But for static electric fields,

$\oint \vec{E} \cdot d\vec{l} = 0$ and $\nabla \times \vec{E} = 0$

whereas for steady magnetic field $\nabla \times \vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H} \cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla.\vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla.\left(\nabla \times \vec{A}\right) = 0$, we can write .

$$\vec{B} = \nabla \times \vec{A}$$

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J}$$

By using vector identity, $\nabla \times \nabla \times \vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2 \vec{A}$

$$\nabla(\nabla.\vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}$$

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$, we get $\nabla^2 \vec{A} = -\mu \vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

.

$$\nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho}{R} dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$

$$\nabla.\vec{A} = \mu\varepsilon \frac{\partial V}{\partial t}$$

In case of time varying fields we shall see that , which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{V'} \frac{J_x}{R} dv'$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv'$$

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$.

Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'} \frac{I}{R} d\vec{l}'$$

$$\vec{A} = \frac{\mu}{4\pi}\int_{S'} \frac{\vec{K}}{R} ds'$$

.

The magnetic flux ψ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s}$$

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A}.d\vec{s} = \oint_C \vec{A}.d\vec{l}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Forces due to magnetic fields

There are three ways in which the force due to magnetic fields can be experienced. The force can be

(a) Force on a charged particle:

We have F_e=QE

This shows that if Q is positive, F_e and E are in same direction. It is found that the magnetic force F_m experienced by a charge Q moving with a velocity u in magnetic field B is

F_m=Qu x B

For a moving change Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$F = F_e+F_m$

or

F=Q(E+u x B)

This is known as Lorentz force equation.

(b) Force on a current element:

To determine the force on a current element Idl of a current carrying conductor due to the magnetic field B, we take the equation

J=P_e u

We have Idl= $\frac{dQ}{dt.}.dl = dQ = \frac{dl}{dt} = dQ.u$

Hence

Idl= dQ.u

This shows that an elemental charge dQ moving with velocity u (thereby producing convection current element dQu) is equivalent to a conduction current element Idl. Thus the force on current element is give by

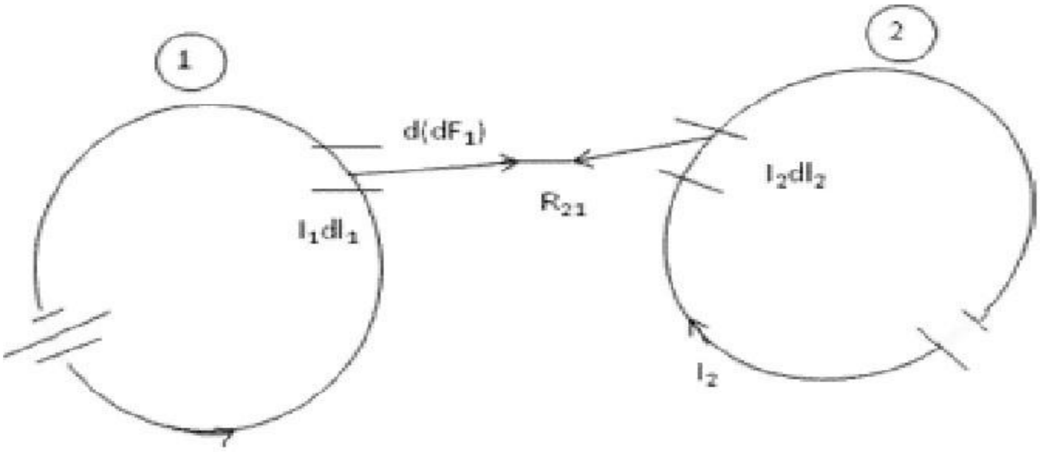
dF= Idl x B

If the current I is through a closed path L or circuit, the force on the circuit is given by

F= $\oint_L Idl \times B$

(c) Force between two current elements:

Consider the force between two elements I_1dl_1 and I_2dl_2. According to biotsavarts law, both current elements produce magnetic fields. Force $d(dF_1)$ on element I_1dl_1 due to field dB_2 produced by element I_2 dl_2 as shown in figure below:



$d(dF_1) = I_1 Dl_1 \times dB_2$

But from biot Savarts law

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2}$$

Hence

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2}$$

This equation is the law of force between two current elements.

We have $F1 = \frac{\mu_0 I_1 I_2 \times a_{R21}}{4\pi} \oint_{L_1} \oint_{L_2} \frac{dI_1 \times (dI_2 \times aR_{21})}{R_{21}^2}$

Inductance:

Inductance is the ability of the material to hold energy in form of magnetic field.

L, I are inductance of material and current flowing in the material.

$$E = \frac{1}{2} LI^2$$

$$\text{Inductance, L} = \frac{\text{Total flux linking current I}}{\text{current (I)}}$$

'B' is induced by I

$$\therefore \phi = \int_S \bar{B}.\bar{ds}$$

Total Flux depends on no of turns

Flux linking for n turns is 'Nϕ'.

$$\boxed{\therefore L = \frac{\lambda}{I}}$$ λ=Nϕ(depending on condition i.e total Flux linking the current)

Inductance of a solenoid:

In the application of ampere's law to solenoid we found that

$$B = \frac{\mu NI}{l} Tesla$$

$$\therefore \phi = B.A = \frac{\mu NIA}{l}$$

With in a loop of N turns, the flux is linking the current N times.

$\therefore$ Total flux linking I = Nϕ

$$= \frac{\mu N^2 IA}{l}$$

$$L = \frac{\lambda}{I} = \frac{\mu N^2 A}{l}$$

Some times inductors are given for unit length as well

$$\therefore \frac{l}{l} = \mu \left(\frac{N}{l} \right)^2 .A$$

Inductance of coaxial cable:

- The total flux linking the inner and outer conductors is same as the flux in the conductor.

$$H = \frac{I}{2\pi r}(A/m)$$

$$B = \frac{\mu I}{2\pi r}(Wb/m^2)$$

Here flux density is differing with radius

$$\therefore \phi = \oint B.d\bar{s}$$

$$\therefore \phi = \int \frac{\mu I}{2\pi r} ds$$

$$d\bar{s} = drdz\phi$$

$$\phi = \int_{z=0}^{L} \int_{r=a}^{b} \frac{\mu I}{2\pi r} drdz$$

$$\phi = \frac{\mu Il}{2\pi} \int_a^b \frac{dr}{r}$$

$$\Rightarrow \lambda = \frac{\mu Il}{2\pi} \ln\left(\frac{b}{a} \right)$$

$$\therefore L = \frac{\lambda}{I} = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a} \right)$$

$$\frac{L}{l} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a} \right)$$

Where μ is the permeability of medium used b/w inner and outer cores.

Also there is current flowing even inside the inner core.

$$= \frac{\mu I l \pi a^2}{8\pi} = \frac{\mu I l}{8\pi}$$

$$\therefore \frac{L_{int}}{l} = \frac{\mu}{8\pi}(H)$$

$$\frac{L_{ext}}{l} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)(H/m)$$

Here μ is permeability of conductor

$$\frac{\text{Total inductance}}{Length} = \frac{L_{ext}}{l} + \frac{L_{int}}{l}$$

$$= \frac{\mu_1}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_2}{2\pi}$$

$$L = \frac{\Lambda}{I} = \frac{N\phi}{I}$$

Magnetic energy or Energy stored in Magnetic Field:

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value I. As mentioned earlier, the self inductance of a coil in general can be written as

$$L = \frac{d\Lambda}{di} = N\frac{d\phi}{di}$$

or

$$L\,di = N\,d\phi$$

If we consider a time varying scenario,

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

We will later see that $N\frac{d\phi}{dt}$ is an induced voltage.

$\therefore v = L\dfrac{di}{dt}$ is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$dW = vi\,dt$$
$$= Li\,di$$

$$W = \int_0^I Li\,di = \frac{1}{2}LI^2 \qquad \text{(Joule)}$$

which is the energy stored in the magnetic circuit.

We can also express the energy stored in the coil in term of field quantities.

For linear magnetic circuit

$$W = \frac{1}{2}\frac{N\phi}{I}I^2 = \frac{1}{2}N\phi I$$

Now,

$$\phi = \int_S \vec{B}.d\vec{S} = BA$$

where A is the area of cross section of the coil. If l is the length of the coil

$$NI = Hl$$

$$\therefore W = \frac{1}{2}HBAl$$

Al is the volume of the coil. Therefore the magnetic energy density i.e., magnetic energy/unit volume is given by

$$W_m = \frac{W}{Al} = \frac{1}{2}BH$$

In vector form

$$W_m = \frac{1}{2}\vec{B}.\vec{H} \qquad \text{J/mt3}$$

is the energy density in the magnetic field.

Module III

TIME VARYING FIELDS

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

(1) $$\nabla \cdot \vec{D} = \rho_v$$

(2) $$\nabla \times \vec{E} = 0$$

For a linear and isotropic medium,

(3) $$\vec{D} = \varepsilon \vec{E}$$

Similarly for the magnetostatic case

(4) $$\nabla . \vec{B} = 0$$

(5) $$\nabla \times \vec{H} = \vec{J}$$

(6) $$\nabla \times \vec{H} = \vec{J}$$

It can be seen that for static case, the electric field vectors $\vec{E}$ $\vec{D}$ and and magnetic field vectors $\vec{B}$ $\vec{H}$ and form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and are therefore not generally introduced in an introductory treatment of the subject, except perhaps as summary relationships.

Faraday's Law of electromagnetic Induction:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

Emf = $-\frac{d\phi}{dt}$ Volts (7)

Where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (7) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

Emf = $-N\frac{d\phi}{dt}$ Volts (8)

By defining the total flux linkage as

$$\lambda = N\phi \qquad (9)$$

The emf can be written as

Emf = $-\frac{d\lambda}{dt}$ (10)

Continuing with equation (3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.\vec{dl}$ (11)

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_S \vec{B}.\vec{ds} \qquad (12)$$

Where S is the surface for which 'C' is the contour.

From (11) and using (12) in (3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}.d\vec{s} \quad 3)$$

By applying stokes theorem

$$\int_s \nabla\times\vec{E}.d\vec{s} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{s} \qquad (14)$$

Therefore, we can write

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad 5)$$

which is the Faraday's law in the point form

$$\frac{d\phi}{dt}$$

We have said that non zero can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf .

Inconsistency of amperes law

Ampere's circuit law states that the line integral of tangential component of H around a closed path is same as the net current Ienc enclosed by the path.

i.e.

$$\int H.dl = I_{enc}$$

By applying stoke's theorem,

$$\int H.dl \; becomes \int_S J.ds$$

$\therefore$ Therefore, $\Delta \times H = J$ ______ (3.14)

This is true in case of static EM fields.

But in case of time-varying fields, the above Ampere's law shows same inconsistency.

The inconsistency of ampere law for time varying fields is shown in two cases:

1. For static EM fields, we have

$$\Delta \times H = J$$

Applying divergence on both sides, we get,

$$\Delta.(\Delta \times H) = \Delta.J$$

But divergence of curl of a vector field is always zero.

Therefore,

$$\Delta.(\Delta \times H) = 0 = \Delta.J$$

The continuity of current equation is given by

$$\Delta.J = \frac{-dp_v}{dt}$$

Where J = Current density

e_v = Charge density

For static fields, no current is produced, therefore, $e_v = 0 \Rightarrow \Delta.J = 0$

Implies eq. 3.15 is satisfied but for time varying fields, current is produced and therefore,

$$\Delta . J = \frac{-de_v}{dt} \# 0 \_\_\_\_\_ \_\_\_\_\_\_\_\_\_\_\_\_ (3.16)$$

Eq. (3.15) and eq. (3.16) are contradicting each other.

This is an inconsistency of ampere's law and the Ampere's law must be modified for time varying fields.

2. Consider the typical example of where the surface passes between the capacitor plates.

The figure is shown below.

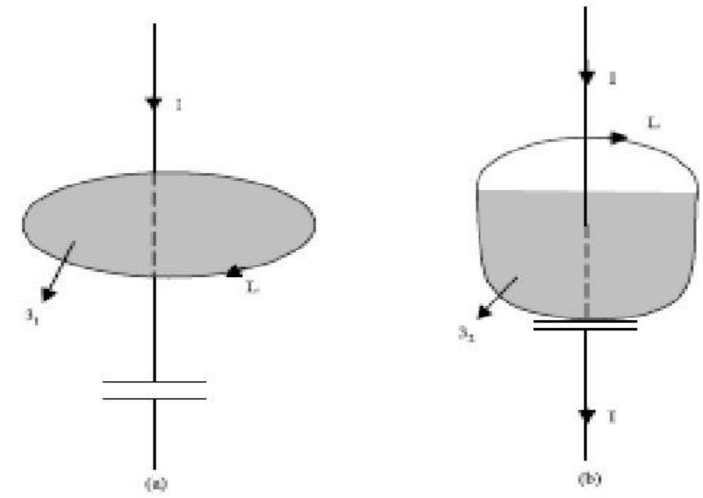


Fig 3.3 (a): Two surfaces of integration which explain the inconsistency of Ampere's law

In fig 3.3(a),

Based on Ampere's circuit law we get figure

$$\int_L H.dl = \int_{S_1} J.ds = I_{enc} = I \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.17)$$

In fig 3.3(b), based the ampere's circuit law, we get,

$$\oint_L H.dl = \int_{3_2} J.ds = I_{enc} = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (3.18)$$

Because no conduction current flows through 3_2

i.e. J=0

in both (a) and (b), same closed path is used, but equations 3.17 and 3.18 are different.

This is an inconsistency of Ampere's circuit law.

This inconsistency of Ampere's circuit law in both cases (1) and (2) can be resolved by including displacement current in Ampere's circuit law.

Substituting in (3.19), we get,

$$\Delta \times H = J + \frac{dD}{dt} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (3.21)$$

This is the Maxwell equation (based on ampere's circuit Law) for tiem varying fields.

In equation (3.21),

J_d = Displacement current density

J = Conduction current density,

The conduction current density J involves flow of charges. The displacement current density J_d does not involve flow of charges. Displacement current,

$$I_d = \int Jd.ds = \int \frac{do}{dt}.ds \text{ \_\_\_\_\_\_\_\_\_\_\_ } (3.22)$$

Displacement Current Density:

The equation

$\Delta \times H = J$ For static EM fields is modified to Modified to

$\Delta \times H = J + J_d$ ________________ (3.19)

To make the Ampere's law compatible for varying fields.

Now, applying divergence, we get

$$\Delta.(\Delta \times H) = 0 = \Delta.J + \Delta J_d$$

$$\Delta.J_d = -\Delta.J = \frac{de_v}{dt}$$

From Gauss Law, we have

$$e_v = \Delta.D$$

Therefore,

$$\Delta.J_d = \frac{d(\Delta.D)}{dt} = \Delta.\frac{dD}{dt}$$

$\Rightarrow J_d = \frac{dD}{dt}$ ____________ (3.20)

Boundary Condition for Magnetic Fields:

Similar to the boundary conditions in the electro static fields, here we will consider the behavior of $\vec{B}$ and $\vec{H}$ at the interface of two different media. In particular, we determine how the tangential and normal components of magnetic fields behave at the boundary of two regions having different permeabilities.

The figure 4.9 shows the interface between two media having permeabities μ_1 and μ_2, $\hat{a}_n$ being the normal vector from medium 2 to medium 1.

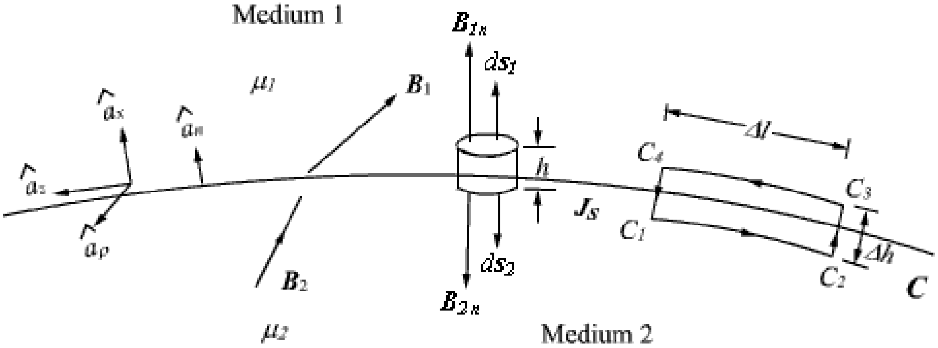


Figure 4.9: Interface between two magnetic media

o determine the condition for the normal component of the flux density vector $\vec{B}$, we consider a small pill box P with vanishingly small thickness *h* and having an elementary area ΔS for the faces. Over the pill box, we can write

$$\oint_S \vec{B}.d\vec{s} = 0 \quad ....(4.36)$$

Since h --> 0, we can neglect the flux through the sidewall of the pill box.

$$\therefore \int_{\Delta S} \vec{B}_1.d\vec{S}_1 + \int_{\Delta S} \vec{B}_2.d\vec{S}_2 = 0 \quad ....(4.37)$$

$$d\vec{S}_1 = dS\hat{a}_n \text{ and } d\vec{S}_2 = dS\left(-\hat{a}_n\right) \quad ....(4.38)$$

$$\therefore \int_{\Delta S} B_{1n} dS - \int_{\Delta S} B_{2n} dS = 0$$

where

Since ΔS is small, we can write

$$\left(B_{1n} - B_{2n}\right)\Delta S = 0$$

or, ... $B_{1n} = B_{2n}$ ....(4.40)

That is, the normal component of the magnetic flux density vector is continuous across the interface.

In vector form,

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \quad \text{...........................(4.41)}$$

To determine the condition for the tangential component for the magnetic field, we consider a closed path C as shown in figure 4.8. By applying Ampere's law we can write

Since h -->0 , $\int_{C_1-C_2} \vec{H}.d\vec{l} + \int_{C_3-C_4} \vec{H}.d\vec{l} = I$

We have shown in figure 4.8, a set of three unit vectors $\hat{a}_n$, $\hat{a}_t$ and $\hat{a}_\rho$ such that they satisfy $\hat{a}_t = \hat{a}_\rho \times \hat{a}_n$ (R.H. rule). Here $\hat{a}_t$ is tangential to the interface and $\hat{a}_\rho$ is the vector perpendicular to the surface enclosed by C at the interface.

$$\oint \vec{H}.d\vec{l} = I$$

if $J_s = 0$, the tangential magnetic field is also continuous. If one of the medium is a perfect conductor J_s exists on the surface of the perfect conductor.

In vector form we can write,

$$\left(\vec{H}_1 - \vec{H}_2\right).\hat{a}_t \Delta l$$

$$= \left(\vec{H}_1 - \vec{H}_2\right).\left(\hat{a}_\rho \times \hat{a}_n\right)\Delta l$$

$$= J_{Sn\Delta l} = \vec{J}_S.\hat{a}_\rho \Delta l$$

Therefore,

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S$$

UNIT – IV

EM Wave Characteristics - I:

- Wave Equations for Conducting and Perfect Dielectric Media
- Uniform Plane Waves - Definition, Relation between E & H
- Wave Propagation in Lossless and Conducting Media
- Wave Propagation in Good Conductors and Good Dielectrics
- Illustrative Problems.

Wave equations:

The Maxwell's equations in the differential form are

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = \vec{\rho}$$

$$\nabla\cdot\vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla\cdot\vec{E} = 0 \qquad (5.29(c))$$

$$\nabla\cdot\vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla\times\nabla\times\vec{A} = \nabla\cdot\left(\nabla\cdot\vec{A}\right) - \nabla^2 A$$

We can write from 2

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}$$

$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Substituting $\nabla\times\vec{H}$ from 1

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free($\nabla\cdot\vec{E} = 0$

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

In the same manner for equation eqn 1

$$\nabla\times\nabla\times\vec{H} = \nabla\cdot\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H}$$

$$= \sigma\left(\nabla\times\vec{E}\right) + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H} = 0$ from eqn 4, we can write

$$\nabla^2\vec{H} = \mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \left(\frac{\partial \vec{H}}{\partial t} \right) + \mu\varepsilon \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

are known as wave equations.

Uniform plane waves:

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wave front or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios.

Let us consider a plane wave which has only E_x component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

The solution to this equation can be written as

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions).

In the time domain, $\varepsilon_X(z,t) = \mathrm{Re}(E_x(z)e^{jwt})$

$$\varepsilon_X(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_X^+(z,t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^+(z,t)$ for several values of t is shown in the Figure below

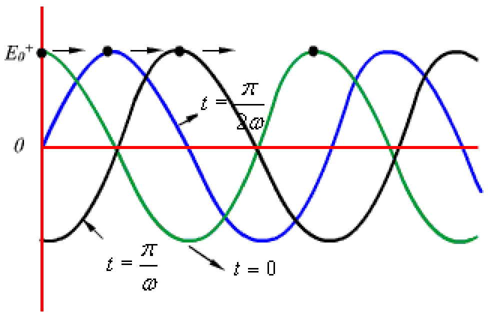


Figure : Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. ,
$\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$$

Or, $\omega \Delta t = k \Delta z$

Or, $\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$

When $\Delta t \to 0$,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity v_p .

$$\therefore v_p = \frac{\omega}{k}$$

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

Then $v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $(\omega t - kz) - \left[\omega t - k(z + \lambda)\right] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \dfrac{\omega}{v_p}$, $\lambda = \dfrac{2\pi v_p}{2\pi f} = \dfrac{v_p}{f}$

or, $\lambda f = v_p$

Thus wavelength λ also represents the distance covered in one oscillation of the wave.

Similarly, $\mathcal{E}^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y$$

where $\eta = \dfrac{\omega\mu}{k} = \dfrac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\dfrac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.

When the wave travels in free space

$\eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$ is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^{+}(z,t) = \hat{a}_y \frac{E_0^+}{\eta}\cos(\omega t - \beta z)$$

Which represents the magnetic field of the wave traveling in the +z direction.

For the negative traveling wave,

$$\vec{H}^{-}(z,t) = -a_y \frac{E_0^+}{\eta}\cos(\omega t + \beta z)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig below

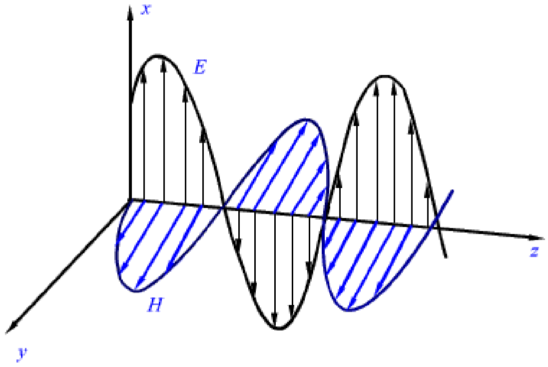


Figure : E & H fields of a particular plane wave at time t.

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$

the above curl equations we can write

$$\nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial\vec{D}}{\partial t} \quad \text{...(1)}$$

In simple medium where $\in$, μ and σ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right) \quad \text{and} \quad \vec{E}.\vec{J} = \sigma E^2$$

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV \quad \text{...........................(2)}$$

The term $\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form ,$e^{j\omega t}$and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\hat{a}_x \quad \text{.................................(1)}$$

when E_0 is real.
Let us consider two instanteneous quantities A and B such that

$$A = \mathrm{Re}\left(Ae^{j\omega t}\right) = |A|\cos(\omega t + \alpha) \quad \text{..................(2)}$$

$$B = \mathrm{Re}\left(Be^{j\omega t}\right) = |B|\cos(\omega t + \beta)$$

where A and B are the phasor quantities.

i.e, $A = |A|e^{j\alpha}$

$B = |B|e^{j\beta}$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right] \quad \text{..............................(3)}$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta)$ ....(4)

Further, considering the phasor quantities *A* and *B*, we find that

$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}(AB^*)$ ....(5)

The poynting vector $\vec{P} = \vec{E} \times \vec{H}$ can be expressed as

$\vec{P} = \hat{a}_x(E_y H_z - E_z H_y) + \hat{a}_y(E_z H_x - E_x H_z) + \hat{a}_z(E_x H_y - E_y H_x)$ ....(6)

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (6.42) we can write:

$\vec{P}_z = E_x(z,t) H_y(z,t) \hat{a}_z$

Using (6)

$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z) H_y^*(z) \hat{a}_z\right)$

$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}(E_x(z) \times H_y(z))$ ....(7)

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.

For a general case, we can write

$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$.....................(8)

We can define a complex Poynting vector

$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}(\vec{S})$.

Polarization of plane wave:

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficient as the magnetic field components are related to electric field vector by the Maxwell's equations.

Let us consider a plane wave travelling in the +z direction. The wave has both E_x and E_y components.

$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$

The corresponding magnetic fields are given by,

$\vec{H} = \frac{1}{\eta}\hat{a}_z \times \vec{E}$

$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right) e^{-j\beta z}$

Depending upon the values of E_{ox} and E_{oy} we can have several possibilities:

1. If E_{oy} = 0, then the wave is linearly polarised in the x-direction.
2. If E_{oy} = 0, then the wave is linearly polarised in the y-direction.
3. If E_{ox} and E_{oy} are both real (or complex with equal phase), once again we get a linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig1 below

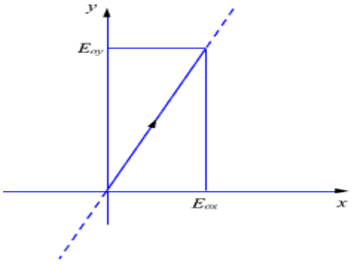


Fig1 : Linear Polarisation

4. If Eox and Eoy are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}|e^{ja}$

$E_{oy} = |E_{oy}|e^{jb}$

Then,

$$E_x(z,t) = \mathrm{Re}\left[|E_{ox}|e^{ja}e^{-j\beta z}e^{j\omega t}\right] = |E_{ox}|\cos(\omega t - \beta z + a)$$

and $E_y(z,t) = \mathrm{Re}\left[|E_{oy}|e^{jb}e^{-j\beta z}e^{j\omega t}\right] = |E_{oy}|\cos(\omega t - \beta z + b)$ ....(2)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (2) we find that,

$$E_x(o,t) = |E_{ox}|\cos\omega t$$

$$E_y(o,t) = |E_{oy}|\cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin\omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2\omega t + \sin^2\omega t = 1$$ ....(3)

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = |E_{ox}|\cos(\omega t)\hat{a}_x - |E_{oy}|\sin(\omega t)\hat{a}_y$$ ....(4)

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 2

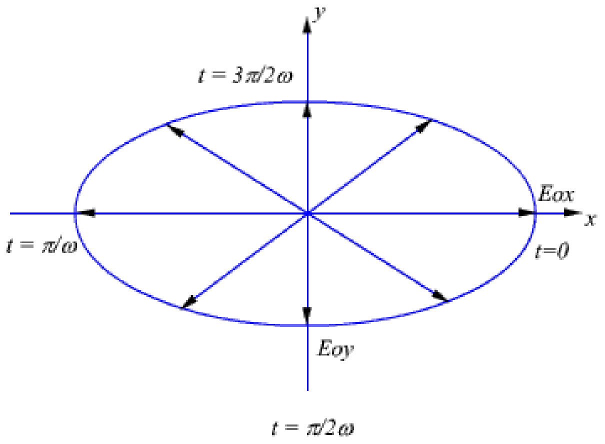


Figure 2 : Plot of *E*(*o*,*t*)

From equation (6.47) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces qn ellipse and the field is said to be elliptically polarised.

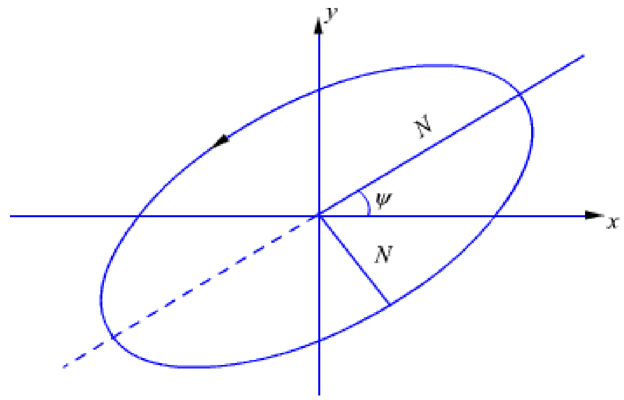


Figure 3: Polarisation ellipse

The polarisation ellipse shown in figure 3 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ(orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW). Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity

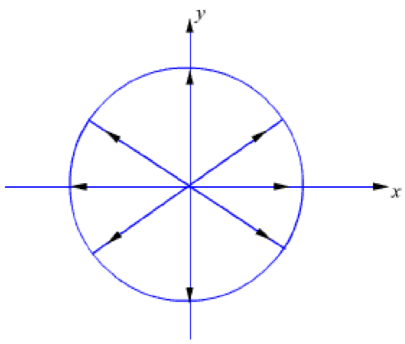


Figure 5: Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

EM Wave Characteristics - II:

- Reflection and Refraction of Plane Waves - Normal for both perfect Conductor and Perfect dielectric
- Brewster Angle
- Critical Angle
- Total Internal Reflection
- Surface Impedance
- Poynting Vector
- Poynting Theorem
- Illustrative Problems.

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

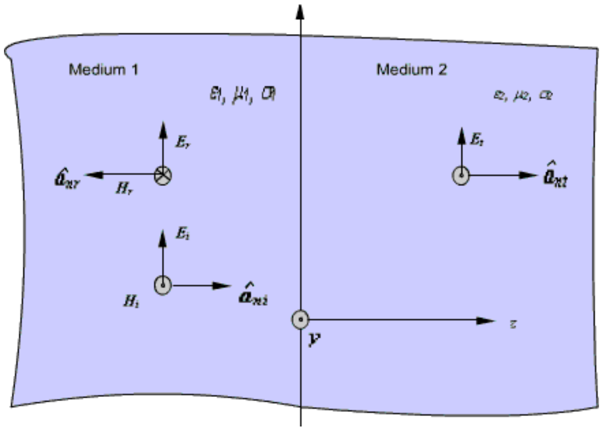


Fig 6 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$. Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively. The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad ....(1)$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad ....(2)$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission.The reflected wave will travel along $\hat{a}_z$ in medium 1.

The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \quad ....(3)$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \quad ....(4)$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \text{ ....(5)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \text{ ....(6)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2,

$\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$

& $\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$

From equation 3to 6 we get,

$$E_{io} + E_{ro} = E_{to} \text{ ....(7)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \text{ ....(8)}$$

Eliminating E_{to},

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}(E_{io} + E_{ro})$$

or, $E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$

or, $E_{ro} = \tau E_{io}$

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \text{ ....(8)}$$

is called the reflection coefficient.

From equation (8), we can write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

or, $E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} \quad \text{.......................................(9)}$$

is called the transmission coefficient.

We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau \quad \text{.......................................(10)}$$

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \leq |\tau| \leq 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$
$$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$$

From (9) and (10)

τ = -1

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

& $\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$ ....(11)

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t \quad \text{..(12)}$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given $'t'$, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

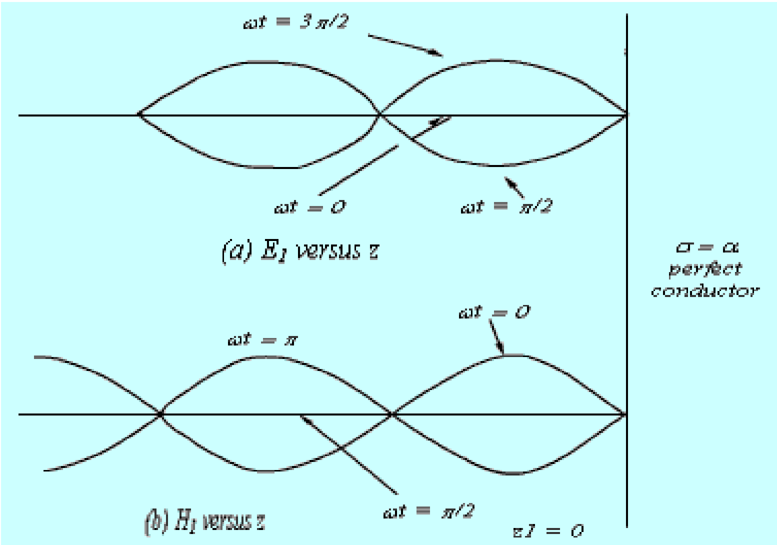


Figure 7: Generation of standing wave

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

$$\left.\right\} \text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4}, \quad n = 0,1,2,\ldots$$

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

Case2: Normal incidence on a plane dielectric boundary : If the medium 2 is not a perfect

conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.

From equation (10) and equation (13) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad \text{..................(14)}$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0,\ \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

In this case both η_1 and η_1 become real numbers.

.................(15)

$$\vec{E}_1 = \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)$$
$$= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)$$
$$= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma\left(2j\sin\beta_1 z\right)\right) \qquad ....(16)$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$. The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows. The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \qquad ....(17)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}(1+T) \qquad ....(18)$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $z_{max} = -\dfrac{n\pi}{\beta_1} = -\dfrac{n\pi}{2\pi/\lambda_1} = -\dfrac{n}{2}\lambda_1$, n = 0, 1, 2, 3.......................(19)

The minimum value of $\left|\vec{E}_1\right|$ is

$$\left|\vec{E}_1\right|_{min} = E_{io}(1-\Gamma) \qquad ....(20)$$

And this occurs when

$$2\beta_1 z_{min} = -(2n+1)\pi$$

or $z_{min} = -(2n+1)\dfrac{\lambda_1}{4}$, n = 0, 1, 2, 3.............................(21)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}(1-\Gamma)$ which occurs at the z_{min} locations and the minimum value of $\left|\vec{E}_1\right|$ is $E_{io}(1+\Gamma)$ which occurs at z_{max} locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\dfrac{|E|_{max}}{|E|_{min}}$ can be written as

$$S = \frac{|E|_{max}}{|E|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad ....(22)$$

The quantity S is called as the standing wave ratio.

As $0 \le |\Gamma| \le 1$ the range of S is given by $1 \le S \le \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z} \left(1 - \Gamma e^{j2\beta_1 z}\right) \quad ....(23)$$

From (6.68) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.

In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface: So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor.

ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

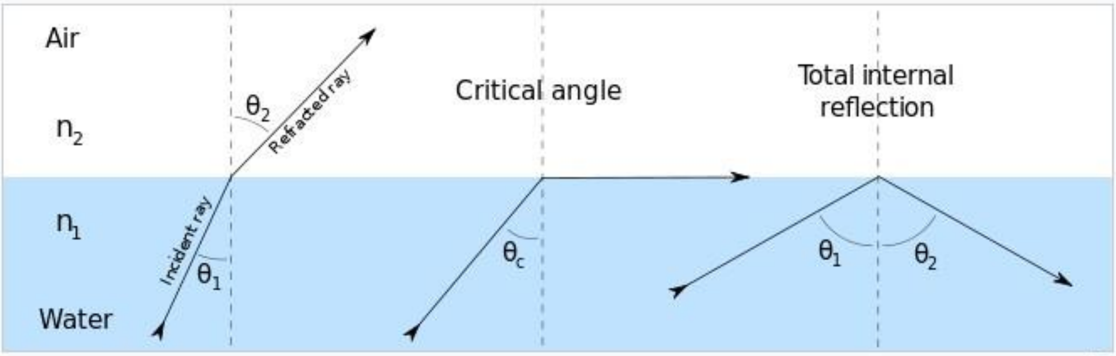
Critical angle:

In geometric optics, at a refractive boundary, the smallest angle of incidence at which total internal reflection occurs. The critical angle is given by

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) ,$$

Where Θ_c is the critical angle, n_1 is the refractive index of the less dense medium, and n_2 is the refractive index of the denser medium.

Angle of incidence: The angle between an incident ray and the normal to a reflecting or refracting surface



Refraction of light at the interface between two media, including total internal reflection.

Brewster's angle

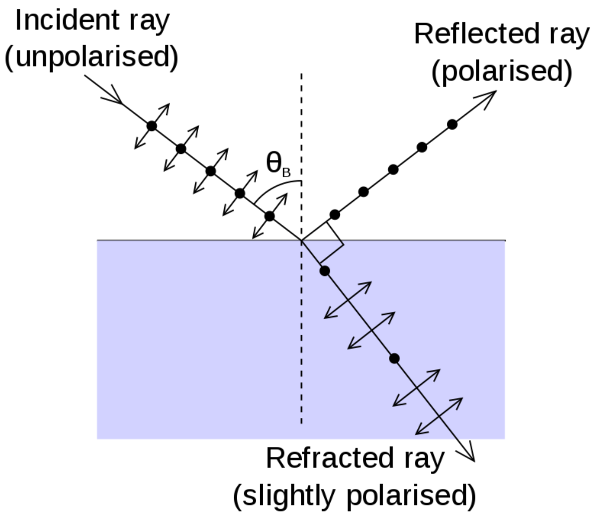
Brewster's angle (also known as the **polarization angle**) is an angle of incidence at which light with a particular polarization is perfectly transmitted through a transparent dielectric surface, with *no reflection*. When *unpolarized* light is incident at this angle, the light that is reflected from the surface is therefore perfectly polarized.

When light encounters a boundary between two media with different refractive indices, some of it is usually reflected as shown in the figure above. The fraction that is reflected is described by the Fresnel equations, and is dependent upon the incoming light's polarization and angle of incidence.

The Fresnel equations predict that light with the *p* polarization (electric field polarized in the same plane as the incident ray and the surface normal at the point of incidence) will not be reflected if the angle of incidence is

$$\theta_B = \tan^{-1}\frac{\eta_2}{\eta_1}$$

where n_1 is the refractive index of the initial medium through which the light propagates (the "incident medium"), and n_2 is the index of the other medium. This equation is known as **Brewster's law**, and the angle defined by it is Brewster's angle.



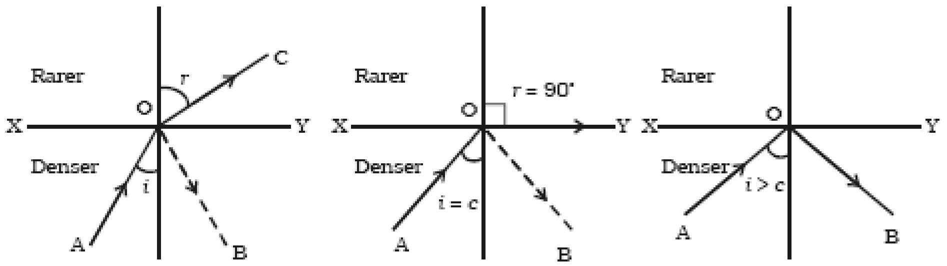
An illustration of the polarization of light that is incident on an interface at Brewster's angle.

Total Internal Reflection

When a ray of light *AO* passes from an optically denser medium to a rarer medium, at the interface *XY*, it is partly reflected back into the same medium along *OB* and partly refracted into the rarer medium along *OC* as shown in figure.

If the angle of incidence is gradually increased, the angle of refraction *r* will also gradually increase and at a certain stage *r* becomes 90^o. Now the refracted ray *OC* is bent so much away from the normal and it grazes the surface of separation of two media. The angle of incidence in the denser medium at which the refracted ray just grazes the surface of separation is called the critical angle c of the denser medium.

If *i* is increased further, refraction is not possible and the incident ray is totally reflected into the same medium itself. This is called *total internal reflection.*



If μ_d is the refractive index of the denser medium then, from Snell's Law, the refractive index of air with respect to the denser medium is given by,

$\mu_a / \mu_d = \sin i / \sin r$

$1/\mu_d = \sin i / \sin r$ (Since, $\mu_a = 1$ for air)

If $r = 90^o$, $i = c$

$\sin c / \sin 90^o = 1/\mu_d$

Or, $\sin c = 1/\mu_d$

Or, $c = \sin^{-1}(1/\mu_d)$

If the denser medium is glass, $c = \sin^{-1}(1/\mu_g)$

Hence for total internal reflection to take place (i) light must travel from a denser medium to a rarer medium and (ii) the angle of incidence inside the denser medium must be greater than the critical angle i.e. $i > c$.